Appendix

$$s_1 = \frac{(C1(R1+R2)+C2R2)+\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}{2\times C1C2R1R2} \quad \text{(Eq. A.1)}$$

$$s_2 = \frac{(C1(R1+R2)+C2R2)-\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}{2\times C1C2R1R2} \quad \text{(Eq. A.2)}$$

$$A' = \frac{-C1R2}{\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.3)}$$

$$B' = \frac{C1R2}{\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.4)}$$

$$C' = \frac{C2R2-C1(R1+R2)-\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}}{-2\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.5)}$$

$$D' = \frac{C2R2-C1(R1+R2)+\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}}{2\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.6)}$$

$$N_1 = \frac{2\times C2R2}{C1(R1+R2)-C2R2+\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.7)}$$

$$N_2 = \frac{2\times C2R2}{C2R2-C1(R1+R2)-\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.8)}$$

$$N_3 = \frac{2\times C2R2}{C1(R1+R2)-C2R2-\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.9)}$$

$$N_4 = \frac{2\times C2R2}{C2R2-C1(R1+R2)+\sqrt{(C1R1-C2R2)^2+C1^2(2R1R2+R2^2)+2C1C2R2^2}} \quad \text{(Eq. A.10)}$$

$$T_1 = V_G\left(1-e^{-\frac{DT}{C1R1}}\right) \quad \text{(Eq. A.11)}$$

$$T_2 = e^{-\frac{DT}{C1R1}} \quad \text{(Eq. A.12)}$$

$$T_3 = \left(A'\times e^{-s_1\times\bar{D}T}+B'\times e^{-s_2\times\bar{D}T}\right) \quad \text{(Eq. A.13)}$$

$$T_4 = e^{-\frac{DT}{C2R2}} \quad \text{(Eq. A.14)}$$

$$T_5 = (C'\times e^{-s_1\times\bar{D}T}+D'\times e^{-s_2\times\bar{D}T}) \quad \text{(Eq. A.15)}$$

$$T_6 = \left(N_2\times A'\times e^{-s_1\times\bar{D}T}+N_4\times B'\times e^{-s_2\times\bar{D}T}\right) \quad \text{(Eq. A.16)}$$

$$T_7 = \left(N_2\times C'\times e^{-s_1\times\bar{D}T}+N_4\times D'\times e^{-s_2\times\bar{D}T}\right) \quad \text{(Eq. A.17)}$$

Derivations

Derivation 1

$$i_1(s)=\frac{V_{C2}^{II}(s)-\frac{V_3}{s}}{R1+\frac{1}{s\times C1}}$$

$$i_2(s)=\left(V_{C2}^{II}(s)-\frac{V_4}{s}\right)s\times C2$$

$$i_3(s)=\frac{V_{C2}^{II}(s)}{R2}$$

$$i_1(s)+i_2(s)+i_3(s)=0$$

$$\frac{V_{C2}^{II}(s)}{R1+\frac{1}{s\times C1}}-\frac{V_3}{s\left(R1+\frac{1}{s\times C1}\right)}+V_{C2}^{II}(s)\times s\times C2-V4\times C2+\frac{V_{C2}^{II}(s)}{R2}=0$$

$$V_{C2}^{II}(s)\times\left(\frac{1}{R1+\frac{1}{s\times C1}}+s\times C2+\frac{1}{R2}\right)=\frac{V3}{s\left(R1+\frac{1}{s\times C1}\right)}+V4\times C2$$

$$V_{C2}^{II}(s)\times\left(\frac{C2R2\left(s^2+s\left(\frac{1}{C2R1}+\frac{1}{C1R1}+\frac{1}{C2R2}\right)+\frac{1}{C1C2R1R2}\right)}{R2\left(s+\frac{1}{C1R1}\right)}\right)=\frac{\left(\frac{V3}{R1}\right)}{\left(s+\frac{1}{C1R1}\right)}+V4\times C2$$

$$V_{C2}^{II}(s)=\frac{\frac{V3}{R1}+V4\times C2\left(s+\frac{1}{C1R1}\right)}{C2\left(s^2+s\left(\frac{1}{C2R1}+\frac{1}{C1R1}+\frac{1}{C2R2}\right)+\frac{1}{C1C2R1R2}\right)}$$

Derivation 2

We now factor the denominator of the final equation in Derivation 1. As a quadratic equation we know it has two roots, but we must show that the roots are real, regardless of the component values selected for C1, C2, R1 and R2. Imaginary roots in our time-domain equation would indicate the presence of sinusoidal signals, which (intuition tells us) should not happen with RC circuits.

$$s^2+s\left(\frac{1}{C2R1}+\frac{1}{C1R1}+\frac{1}{C2R2}\right)+\frac{1}{C1C2R1R2}=0$$

$$s^2+s\left(\frac{C1R2+C2R2+C1R1}{C1C2R1R2}\right)+\frac{1}{C1C2R1R2}=0$$

$$s_{1,2}=\frac{-\left(\frac{C1(R1+R2)+C2R2}{C1C2R1R2}\right)\pm\sqrt{\left(\frac{C1(R1+R2)+C2R2}{C1C2R1R2}\right)^2-\frac{4}{C1C2R1R2}}}{2}$$

Next, we concentrate on the term inside the square root sign, and make sure that it is a positive number, which in turn guarantees real roots and no sinusoids at the output:

$$\left(\frac{C1(R1+R2)+C2R2}{C1C2R1R2}\right)^2-\frac{4}{C1C2R1R2}=\frac{C1^2(R1+R2)^2+2\times C1(R1+R2)C2R2+C2^2R2^2-4C1C2R1R2}{(C1C2R1R2)^2}=$$

$$\frac{C1^2(R1^2+2R1R2+R2^2)+2C1C2R1R2+2C1C2R2^2+C2^2R2^2-4\times C1C2R1R2}{(C1C2R1R2)^2}=$$

$$\frac{C1^2R1^2+2\times R1R2C1^2+C1^2R2^2+2\times C1C2R1R2+2\times C1C2R2^2+C2^2R2^2-4\times C1C2R1R2}{(C1C2R1R2)^2}$$

$$\frac{C1^2R1^2+2\times R1R2C1^2+C1^2R2^2-2C1C2R1R2+2\times C1C2R2^2+C2^2R2^2}{(C1C2R1R2)^2}$$

Rearranging the terms of the numerator we get:

$$\frac{(C1^2R1^2-2\times C1C2R1R2+C2^2R2^2)+2\times R1R2C1^2+C1^2R2^2+2\times C1C2R2^2}{(C1C2R1R2)^2}=$$

$$\frac{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R}{(C1C2R1R2)^2}$$

We see from the last equation that the quantity inside the square root is positive, so the roots are indeed real, and correspond to exponential functions in the time domain. We can see this more clearly after factoring them. Going back to the equation for roots s_1 and s_2, we get:

$$s_{1,2}=\frac{-\left(\frac{C1(R1+R2)+C2R2}{C1C2R1R2}\right)\pm\sqrt{\frac{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}{(C1C2R1R2)^2}}}{2}$$

$$s_{1,2}=\frac{-(C1(R1+R2)+C2R2)\pm\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}{2\times C1C2R1R2}$$

$$s_1=\frac{(C1(R1+R2)+C2R2)+\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}{2\times C1C2R1R2}$$

$$s_2=\frac{(C1(R1+R2)+C2R2)-\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}{2\times C1C2R1R2}$$

Note that the minus sign has been omitted from s_1 and s_2, because we want to represent this quadratic equation in terms of its roots, in the form $(s + s_1)(s + s_2)$.

Derivation 3

$$V_{C2}^{II}(s)=\frac{\left(\frac{V3}{C2R1}\right)}{(s+s_1)(s+s_2)}+\frac{V4\left(s+\frac{1}{C1R1}\right)}{(s+s_1)(s+s_2)}$$

$$V_{C2}^{II}(s)=V3\left(\frac{A'}{s+s_1}+\frac{B'}{s+s_2}\right)+V4\left(\frac{C'}{s+s_1}+\frac{D'}{s+s_2}\right)$$

$$V_{C2}^{II}(s)=\frac{(V3\times A'+V4\times C')}{(s+s_1)}+\frac{(V3\times B'+V4\times D')}{(s+s_2)}$$

where:

$$A' = \left.\frac{\left(\frac{1}{C2R1}\right)}{s+s_2}\right|_{s=-s_1} = \frac{-C1R2}{\sqrt{(C1R1-C2R2)^2 + C1^2(2\times R1R2+R2)^2 + 2\times C1C2R2^2}}$$

$$B' = \left.\frac{\left(\frac{1}{C2R1}\right)}{s+s_1}\right|_{s=-s_2} = \frac{C1R2}{\sqrt{(C1R1-C2R2)^2 + C1^2(2\times R1R2+R2^2) + 2\times C1C2R2^2}}$$

$$C' = \left.\frac{s+\frac{1}{C1R1}}{s+s_2}\right|_{s=-s_1} = \frac{C2R2 - C1(R1+R2) - \sqrt{(C1R1-C2R2)^2 + C1^2(2\times R1R2+R2^2) + 2\times C1C2R2^2}}{-2\sqrt{(C1R1-C2R2)^2 + C1^2(2\times R1R2+R2^2) + 2\times C1C2R2^2}}$$

$$D' = \left.\frac{s+\frac{1}{C1R1}}{s+s_1}\right|_{s=-s_2} = \frac{C2R2 - C1(R1+R2) + \sqrt{(C1R1-C2R2)^2 + C1^2(2R1R2+R2^2) + 2C1C2R2^2}}{2\sqrt{(C1R1-C2R2)^2 + C1^2(2R1R2+R2^2) + 2C1C2R2^2}}$$

$$V_{II}(t) = (V3\times A' + V4\times C')e^{-s_1\times(t)} + (V3\times B' + V4\times D')e^{-s_2\times(t)}$$

Derivation 4

$$V_{C1}^{II}(s) = i_1(s)\times\frac{1}{s\times C1} + \frac{V3}{s}$$

$$V_{C1}^{II}(s) = \left(\frac{V_{C2}^{II}(s)}{R1}\times\frac{s}{s+\frac{1}{C1R1}} - \frac{\left(\frac{V3}{R1}\right)}{s+\frac{1}{C1R1}}\right)\times\frac{1}{s\times C1} + \frac{V3}{s}$$

$$V_{C1}^{II}(s) = \frac{V_{C2}^{II}(s)}{C1R1}\times\frac{1}{\frac{1}{C1R1}} - \frac{V3}{C1R1}\times\frac{1}{s\left(s+\frac{1}{C1R1}\right)} + \frac{V3}{s}$$

Derivation 5

$$V_{CI}^{II}(s) = \left(\frac{(V3\times A' + V4\times C')}{(s+s_1)} + \frac{(V3\times B' + V4\times D')}{(s+s_2)}\right)\times\frac{\left(\frac{1}{C1R1}\right)}{s+\frac{1}{C1R1}} - V3\left(\frac{\left(\frac{1}{C1R1}\right)}{s\left(s+\frac{1}{C1R1}\right)}\right) + \frac{V3}{s}$$

$$V_{C1}^{II}(s) = \left(\frac{(V3\times A' + V4\times C')\left(\frac{1}{C1R1}\right)}{(s+s_1)\left(s+\frac{1}{C1R1}\right)} + \frac{(V3\times B' + V4\times D')\left(\frac{1}{C1R1}\right)}{(s+s_2)\left(s+\frac{1}{C1R1}\right)}\right) - V3\left(\frac{Z1}{s} + \frac{Z2}{s+\frac{1}{C1R1}}\right) + \frac{V3}{s}$$

$$V_{C1}^{II}(s) = \Bigg((V3\times A' + V4\times C')\left(\frac{N1}{\left(s+\frac{1}{C1R1}\right)} + \frac{N2}{(s+s_1)}\right) + (V3\times B' + V4\times D')\left(\frac{N3}{\left(s+\frac{1}{C1R1}\right)} + \frac{N4}{(s+s_2)}\right) -$$

$$V3\left(\frac{Z1}{s} + \frac{Z2}{s+\frac{1}{C1R1}}\right) + \frac{V3}{s}\Bigg)$$

$$V_{C1}^{II}(s)=\left(\frac{(N1(V3\times A'+V4\times C')+N3(V3\times B'+V4\times D')-V3\times Z2)}{\left(s+\frac{1}{C1R1}\right)}+\frac{N2(V3\times A'+V4\times C')}{(s+s_1)}+\right.$$

$$\left.\frac{N4(V3\times B'+V4\times D')}{(s+s_1)}+\frac{V3(1-Z1)}{s}\right)$$

$$N1=\left.\frac{\left(\frac{1}{C1R1}\right)}{(s+s_1)}\right|_{s=-\frac{1}{C1R1}}=\frac{2\times C2R2}{C1(R1+R2)-C2R2+\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}$$

$$N2=\left.\frac{\left(\frac{1}{C1R1}\right)}{\left(s+\frac{1}{C1R1}\right)}\right|_{s=-s_1}=\frac{2\times C2R2}{C2R2-C1(R1+R2)-\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}$$

$$N3=\left.\frac{\left(\frac{1}{C1R1}\right)}{(s+s_2)}\right|_{s=-\frac{1}{C1R1}}=\frac{2\times C2R2}{C1(R1+R2)-C2R2-\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}$$

$$N4=\left.\frac{\left(\frac{1}{C1R1}\right)}{\left(s+\frac{1}{C1R1}\right)}\right|_{s=-s_2}=\frac{2\times C2R2}{C2R2-C1(R1+R2)+\sqrt{(C1R1-C2R2)^2+C1^2(2\times R1R2+R2^2)+2\times C1C2R2^2}}$$

$$Z1=\left.\frac{\left(\frac{1}{C1R1}\right)}{s+\frac{1}{C1R1}}\right|_{s=0}=1$$

$$Z2=\left.\frac{\left(\frac{1}{C1R1}\right)}{s}\right|_{s=-\frac{1}{C1R1}}=-1$$

$$V_{C1}^{II}(s)=\frac{(N1(V3\times A'+V4\times C')+N3(V3\times B'+V4\times D')+V3)}{\left(s+\frac{1}{C1R1}\right)}+\frac{(N2(V3\times A'+V4\times C')}{(s+s_1)}+\frac{N4(V3\times B'+V4\times D')}{(s+s_2)}$$

However, the numerator of the first term in the above equation is equal to zero and thus the voltage of the fly capacitor becomes:

$$V_{C1}^{II}(s)=\frac{N2(V3\times A'+V4\times C')}{(s+s_1)}+\frac{N4(V3\times B'+V4\times D')}{(s+s_2)}$$

$$V_{C1}^{II}(t)=N2(V3\times A'+V4\times C')e^{-s_1\times(t)}+N4(V3\times B'+V4\times D')e^{-s_2\times(t)}$$

Derivation 6

$$\frac{1}{T}\int_0^{DT} V_{C2}^{I}(t)dt = \frac{1}{T}\int_0^{DT} V2\times e^{-\frac{t}{C2R2}}dt = \frac{C2R2V2}{T}\left(1-e^{-\frac{DT}{C2R2}}\right)$$

$$\frac{1}{T}\int_0^{\bar{D}T} V_{II}(t)dt = \frac{1}{T}\int_0^{\bar{D}T}\left((V3\times A'+V4\times C')e^{-s_1\times(t)}+(V3\times B'+V4\times D')e^{-s_2(t)}\right)dt =$$
$$\frac{(V3\times A'+V4\times C')}{s_1+T}\left(1-e^{-s_1\times\bar{D}T}\right)+\frac{(V3\times B'+V4\times D')}{s_2\times T}\left(1-e^{-s_2\times\bar{D}T}\right)$$

Derivation 7

$$V_{C1_{AVERAGE}} = \langle v_{C1}(t)\rangle = \frac{1}{T}\int_0^{T} V_{C1}^{I}(t)dt = \frac{1}{T}\int_0^{\bar{D}T} V_{C1}^{II}(t)dt$$

$$V_{C1_{AVERAGE}} = \frac{1}{T}\int_0^{DT}\left(V_G\left(1-e^{-\frac{t}{C1R1}}\right)+V1\times e^{-\frac{t}{C1R1}}\right)dt +$$
$$\frac{1}{T}\int_0^{\bar{D}T}\left(N2(V3\times A'+V4\times C')e^{-s_2\times(t)}+N4(V3\times B'+V4\times D')e^{-s_2\times(t)}\right)dt =$$
$$\frac{V_G DT + C1R1(V1-V_G)\left(1-e^{-\frac{DT}{C1R1}}\right)}{T}+$$
$$\frac{1}{T}\left[\frac{N2(V3\times A'+V4\times C')}{s_1}(1-e^{-s_1\times\bar{D}T})+\frac{N4(V3\times B'+V4\times D')}{s_2}(1-e^{-s_2\times\bar{D}T})\right]$$

Derivation 8

$$\int_0^{DT}(V_{C2}^{I}(t))^2dt = \int_0^{DT}\left(V2\times e^{-\frac{t}{C2R2}}\right)^2dt = \int_0^{DT} V2^2\times e^{-\frac{2\times t}{C2R2}}dt = \frac{V2^2\,C2R2}{2}\left(1-e^{-\frac{2\times DT}{C2R2}}\right)$$

$$\int_0^{\bar{D}T}\left(V_{C2}^{II}(t)\right)^2dt = \int_0^{\bar{D}T}\left((V3\times A'+V4\times C')e^{-s_1\times(t)}+(V3\times B'+V4\times D')e^{-s_2\times(t)}\right)^2dt =$$

$$\frac{(V3\times A'+V4\times C')^2}{2\times s_1}(1-e^{-2\times s_1\times\bar{D}T})+\frac{2(V3\times A'+V4\times C')(V3\times B'+V4\times D')}{(s_1+s_2)}(1-e^{-(s_1+s_2)\bar{D}T})+$$
$$\frac{(V3\times B'+V4\times D')^2}{2\times s_2}(1-e^{-2\times s_2\times\bar{D}T})$$